

# GLOBAL GENERATION OF ADJOINT LINE BUNDLES ON PROJECTIVE 5-FOLDS

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**ABSTRACT.** Let  $X$  be a smooth projective variety of dimension 5 and  $L$  be an ample line bundle on  $X$  such that  $L^5 > 7^5$  and  $L^d \cdot Z \geq 7^d$  for any subvariety  $Z$  of dimension  $1 \leq d \leq 4$ . We show that  $\mathcal{O}_X(K_X + L)$  is globally generated.

## 1. INTRODUCTION

Throughout this paper, we work over an algebraic closed field  $k$  of characteristic zero.

Geometric properties of pluricanonical and adjoint line bundles on surfaces and higher dimensional varieties have been extensively studied. Motivated by work of Kodaira [Kod68] and Bombieri [Bom73], who studied pluricanonical maps of surfaces of general type, one wants to understand explicitly when the pluricanonical line bundles or more generally adjoint line bundle on higher dimension is globally generated or very ample. In 1985, Fujita [Fuj87] raised the following conjecture.

**Conjecture 1.1** (Fujita). *Let  $X$  be a smooth projective variety of dimension  $n$  and  $H$  be an ample divisor on  $X$ . Then  $\mathcal{O}_X(K_X + mH)$  is globally generated if  $m \geq n + 1$ .  $\mathcal{O}_X(K_X + mH)$  is very ample if  $m \geq n + 2$ .*

For curves, the conjecture follows from Riemann-Roch theorem. For surfaces, the conjecture was proved by Reider [Rei88] using Bogomolov's instability theorem [Bog78] on rank two vector bundles. Unfortunately, this approach is limited to surfaces because no generalization of Bogomolov's theorem in higher dimension was available.

In higher dimensions, the first effective result was obtained by Demailly [Dem93] using analytic tools. Given an ample line bundle  $H$ , Demailly proved that  $\mathcal{O}_X(2K_X + 12n^n H)$  is very ample. Kollár [Kol93] then proved that  $\mathcal{O}_X(2(n+1)(n+2)!(K_X + (n+2)H))$  is globally generated following a cohomological approach developed by Kawamata [Kaw84], Reid [Rei83] and Shokurov [Sho85].

Roughly speaking, given a point  $x \in X$ , to show that  $\mathcal{O}_X(K_X + L)$  is globally generated at  $x$ , the idea is to create an effective  $\mathbb{Q}$ -divisor  $D$  linearly equivalent to  $\lambda L$ ,  $0 \leq \lambda < 1$ , so that the support  $Z(D)$  of the multiplier ideal sheaf of  $D$  is

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a normal subscheme containing  $x$ . Then one applies Kawamata-Viehweg vanishing theorem to reduce the problem to the subscheme  $Z(D)$ . However, a major difficulty is that the subscheme  $Z(D)$  in general could be highly singular.

When  $X$  is a three dimensional smooth algebraic variety, one can choose  $D$  so that  $Z(D)$  has at most Kawamata log terminal (klt for short) singularities. By showing a Reider type theorem on an algebraic surface with klt singularities, Ein and Lazarsfeld [EL93] proved, among others, that Fujita's freeness conjecture is true for  $n = 3$ . Using the same method with carefully estimating the restrict volume of the pullback of the ample line bundle  $L$  along the exception divisor  $E$  of the blowing up at  $x \in X$ , Fujita [Fuj93] improved Ein and Lazarsfeld's results.

By applying a theorem on extension of  $L^2$  holomorphic functions, Angehrn and Siu [AS95], overcome the difficulty on controlling singularities of  $Z(G)$  and lowered the bound to a quadratic bound.

**Theorem 1.2** (Angehrn-Siu). *Let  $X$  be a smooth projective variety of dimension  $n$  and  $L$  be an ample line bundle. If  $m \geq \frac{1}{2}n(n+1) + 1$ , then  $\mathcal{O}_X(K_X + mL)$  is globally generated.*

The method was later translated into algebraic setting by Kollár [Kol97]. Following the work of Fujita, and Angehrn and Siu, Helmke [Hel99] obtained effective bounds in arbitrary dimensions in the spirit of Reider.

**Theorem 1.3** (Helmke). *Let  $X$  be a smooth projective variety of dimension  $n$ ,  $x \in X$  be a point and  $L$  be an ample line bundle satisfying the following conditions*

- (1)  $L^n > n^n$ ;
- (2)  $L^{n-1} \cdot H \geq n^{n-1}$  for all hypersurface  $H \subset X$  such that  $x \in H$ ;
- (3)  $L^d \cdot Z \geq \text{mult}_x Z \cdot n^d$  for all  $Z \subset X$  such that  $x \in Z$ , where  $d = \dim Z < n - 1$ .

*Then  $\mathcal{O}_X(K_X + mL)$  is globally generated at  $x$ .*

Helmke also showed that  $\text{mult}_x Z \leq \binom{n-1}{d-1}$  which makes the bound better than Angehrn and Siu's when dimension is not too large. Balancing both results, Heier [Hei02] obtain the following.

**Theorem 1.4** (Heier). *Let  $X$  be a smooth projective variety of dimension  $n$  and  $L$  be an ample line bundle. If  $m \geq (e + \frac{1}{2})n^{\frac{4}{3}} + n^{\frac{2}{3}} + 1$ , then  $\mathcal{O}_X(K_X + mL)$  is globally generated.*

Helmke's result together with those of Reider, Ein and Lazarsfeld, and Fujita leads to the following stronger conjecture.

**Conjecture 1.5.** *Let  $X$  be a smooth projective variety of dimension  $n$ ,  $x \in X$  be a point and  $L$  be an ample line bundle. Assume that  $L^n > n^n$  and  $L^d \cdot Z \geq n^d$  for all  $Z \subset X$  with  $x \in Z$ , where  $d = \dim Z < n$ . Then  $\mathcal{O}_X(K_X + mL)$  is globally generated at  $x$ .*

Conjecture 1.5 is true for surfaces and 3-folds. When  $n = 4$ , the following theorem of Kawamata [Kaw97] confirmed Fujita's base point freeness conjecture but not Conjecture 1.5.

**Theorem 1.6** ([Kaw97]). *Let  $X$  be a smooth projective variety of dimension 4,  $L$  be an ample line bundle on  $X$  and  $x \in X$  be a point. Assume that  $L^d \cdot Z \geq 5^d$  for all subvariety  $Z \subseteq X$  of dimension  $d$  which contains  $x$ . Then  $\mathcal{O}_X(K_X + L)$  is globally generated at  $x$ .*

In this paper, we show a Kawamata-type result on projective 5-folds by carefully analyzing upper bounds of deficit functions (Definition 2.4) and applying Helmke's induction criterion (Proposition 2.7). More precisely, we prove that  $\mathcal{O}_X(K_X + L)$  is globally generated if  $\sqrt[5]{L^5} > 7$  and  $\sqrt[d]{L^d \cdot Z} \geq 7$  for any subvariety  $Z$  of dimension  $d \leq 4$  (Theorem 4.3). The same proof also works on 4-folds and reproduces Kawamata's result.

We start by recalling some definitions and results from [Hel99] and [Ein97] in Section 2. In section 3, we study restricted volumes along exceptional divisors and their applications. We prove our result in Section 4.

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## 2. THE DEFICIT FUNCTION AND CRITICAL VARIETIES

In this section,  $X$  will be a smooth projective variety of dimension  $n$  and  $G$  will be an effective  $\mathbb{Q}$ -divisor on  $X$ . The multiplier ideal of  $G$  is defined as  $\mathcal{I}(G) = f_*\mathcal{O}_Y(K_{Y/X} - [f^*G])$ , where  $f : Y \rightarrow X$  is a log resolution of  $G$ . We denote by  $Z(G)$  the scheme defined by  $\mathcal{I}(G)$ .

To prove the global generation of an adjoint line bundle. The key observation is the following lemma.

**Lemma 2.1.** *Assume that  $G$  is an effective  $\mathbb{Q}$ -divisor such that  $Z(G)$  is 0-dimensional at  $x$ . Let  $A$  be an integral divisor such that  $A - (K_X + G)$  is nef and big. Then the line bundle  $\mathcal{O}_X(A)$  is globally generated at  $x$ .*

To apply this lemma to the adjoint line bundle  $K_X + L$ , we will assume that  $L^n > n^n$ . By Riemann-Roch theorem, we can construct an effective divisor  $G$  linearly equivalent to  $\lambda L$  for some  $0 < \lambda < 1$  such that  $x \in Z(G)$ . Then we will show that  $\dim Z(G) = 0$  for a suitable choice of  $G$ . However, for a general choice of  $G$ ,  $\dim Z(G)$  may be positive, we have to find a way to modify the initial divisor  $G$  to reduce the dimension. For that, we want  $Z(G)$  to be "minimal" in the following sense.

**Definition 2.2.** Let  $x$  be a point in  $X$  which is contained in  $Z(G)$ . We say that  $G$  is *critical* at  $x$  if :

- (1)  $x \notin Z((1 - \varepsilon)G)$  for any  $0 < \varepsilon < 1$ ;
- (2)  $K_{Y/X} - [f^*G] = P - F - N$  for a log resolution  $f : Y \rightarrow X$  such that  $P$ ,  $F$  and  $N$  are effective with no common component and  $F$  is irreducible and  $N \cap f^{-1}(x) = \emptyset$ .

The component  $F$  is called the critical component of  $G$  at  $x$  and  $f(F) = Z(G)$  is called the *critical variety* of  $G$  at  $x$ .

*Remark 2.3* (Existence, minimality and normality of critical varieties). The technical detail of the following remarks can be found in [Ein97] or [Lee99].

- (1) In general, an effective divisor  $G$  may not satisfy the condition 2.2 (2). However, if  $G$  is ample, then by perturbing  $G$  a little bit, one can construct a new divisor  $G'$  linearly equivalent to  $(1 + \varepsilon)G$  with  $0 \leq |\varepsilon| \ll 1$  such that  $G'$  is critical at  $x$ . This technique is called the tie-breaking trick.
- (2) Assume that  $G$  is critical. Then  $(X, G)$  is log canonical at the point  $x$  with the minimal log canonical center  $Z(G)$ . Conversely, assume that  $(X, G)$  is log canonical at  $x$  with the minimal log canonical center  $Z$ . If  $G$  is not critical at  $x$ , then by the tie-breaking trick, one can create a new divisor  $G'$  linearly equivalent to  $(1 + \varepsilon)G$  with  $0 \leq |\varepsilon| \ll 1$  such that  $G'$  is critical at  $x$  with the critical variety  $Z(G') = Z$ .
- (3) Let  $Z$  be a critical variety at  $x$ . Then  $Z$  is normal at  $x$ . In particular, if  $Z$  is a curve, then  $Z$  is smooth at  $x$ .

An invariant measuring the difficulty to find a new divisor  $G'$  with  $Z(G') \subsetneq Z(G)$  is the deficit  $\text{def}_x(G)$  of  $G$  at  $x$ . This concept was introduced by Ein and Helmke independently (see [Ein97] and [Hel97]). We follow Ein's definition (Section 4, [Ein97]) with adaptation of Helmke's definition of "wildness" (Section 3, [Hel99]).

Let  $\pi : Y' \rightarrow X$  be the blowing up of  $x$  and  $E$  be the exceptional divisor. Let  $g : Y \rightarrow Y'$  be a log resolution of  $\pi^*(G) + E$ . We then have a log resolution  $f = g \circ \pi : Y \rightarrow X$ . Write  $K_Y - f^*(K_X + G) = \sum a_j F_j$  and  $g^*(E) = e_j F_j$ .

**Definition 2.4.** If  $x \notin Z(G)$ , we define the *deficit* of  $G$  at  $x$  as

$$\text{def}_x(G) := \inf_{f(F_j)=x} \left\{ \frac{a_j + 1}{e_j} \right\};$$

if  $G$  is log canonical at  $x$ , we define

$$\text{def}_x(G) := \lim_{t \rightarrow 0^+} \text{def}_x((1 - t)G).$$

Let  $Z$  be a subvariety of  $X$  containing  $x$  and  $G$  be an effective divisor such that  $(X, G)$  is log canonical at  $x$ . We define the *relative deficit* of  $G$  over  $Z$  at  $x$  as

$$\text{def}_x^Z(G) := \sup_{D \geq 0} \{ \text{def}_x(G + D) \mid (X, G + D) \text{ is log canonical at } x, Z(G + D) = Z \}.$$

If there is no effective divisor  $D$  such that  $(X, G + D)$  is log canonical at  $x$  with  $Z(G + D) = Z$ , we define  $\text{def}_x^Z(G) = 0$ . In the case that  $G = \emptyset$ , we write  $\text{def}_x^Z$  for  $\text{def}_x^Z(\emptyset)$ .

*Remark 2.5.* Assume that  $G$  is log canonical at  $x$ . One notes that the deficit is the same as Helmke's local discrepancy, i.e.

$$\text{def}_x(G) = \sup\{\text{ord}_x D \mid (X, G + D) \text{ is log canonical at } x \text{ for all } \mathbb{Q}\text{-effective } D\}.$$

From this interpretation, we see that the definition of deficit is independent of the choice of log resolution. If  $Z = Z(G)$ , then  $\text{def}_x^Z(G) = \text{def}_x(G)$ .

The following lemma, based on Siu's idea, shows when we can construct a new divisor  $G'$  with  $Z(G') \subsetneq Z(G)$ .

**Lemma 2.6** ([Ein97]). *Let  $X$  be a smooth projective variety with an ample line bundle  $L$ . Let  $G$  be an effective  $\mathbb{Q}$ -divisor on  $X$  with critical variety  $Z$  at  $x \in X$  and  $B$  an effective  $\mathbb{Q}$ -divisor on  $Z$ , which is linearly equivalent to  $qL|_Z$  for some positive rational number  $q$ . Assume that  $\text{ord}_x B > \text{def}_x(G)$ . Then there is an effective  $\mathbb{Q}$ -divisor  $D$  linearly equivalent to  $qL$  on  $X$  such that  $D|_Z = B$ , and the new divisor  $G' = (1 - t)G + D'$  for a sufficiently small positive number  $t$  such that  $G'$  is critical at  $x$  with  $Z((1 - t)G + D') \subsetneq Z$ , where  $D'$  is a small perturbation of  $sD$  for some  $0 < s \leq 1$ , more precisely,  $D' = s'D + \delta H$  with  $s - s'$  and  $\delta$  sufficiently small positive numbers and  $H \in |L|$  a divisor passing through  $x$ . Moreover,*

$$\text{def}_x(G') \leq \text{def}_x((1 - t)G) - \text{ord}_x D'|_Z.$$

In practical, we have the following induction criterion due to Helmke which can be viewed as a consequence of the above lemma.

**Proposition 2.7** ([Hel97]). *Let  $X$  be a smooth projective variety,  $L$  be an ample line bundle over  $X$  and  $G$  be a  $\mathbb{Q}$ -divisor linearly equivalent to  $\lambda L$  for some positive rational number  $\lambda < 1$ . Assume that  $G$  is critical at  $x$  with  $\text{def}_x(G)$ . Let  $Z$  be the critical variety of  $G$  at  $x$  and  $d = \dim Z > 0$ . If*

$$(1) \quad L^d \cdot Z > \left(\frac{\text{def}_x(G)}{1 - \lambda}\right)^d \cdot \text{mult}_x Z$$

*then there is a  $\mathbb{Q}$ -divisor  $G'$  linearly equivalent to  $\lambda' L$  with  $\lambda < \lambda' < 1$  such that  $G'$  is critical at  $x$  with the critical variety  $Z'$  which is properly contained in  $Z$  and*

$$\frac{\text{def}_x(G')}{1 - \lambda'} < \frac{\text{def}_x(G)}{1 - \lambda}.$$

Now it is clear how useful upper bounds for the deficit  $\text{def}_x(G)$  are. Thank to Ein and Helmke, we already have some upper bounds on  $\text{def}_x(G)$ .

**Proposition 2.8** ([Ein97], [Hel99]).

$$(1) \quad \text{def}_x(G) \leq n - \text{ord}_x G.$$

- (2) If  $G$  is critical at  $x$ , then the critical variety is of dimension zero at  $x$  if and only if  $\text{def}_x(G) = 0$ .
- (3) Assume that  $G$  is critical at  $x$  and  $x \in Z$  where  $Z$  is a subvariety of  $X$ . Let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $Z$  is not in the support of  $D$  and  $\text{def}_x^Z(G + D) > 0$ . Then  $\text{def}_x^Z(G + D) \leq \text{def}_x^Z(G) - \text{ord}_x D$ .

By Proposition 2.8 (1) and the definition of relative deficit, we know that if the critical variety  $Z$  of  $G$  is a hypersurface, then

$$\text{def}_x^Z = \text{def}_x^Z(Z) = \text{def}_x(Z) \leq n - \text{ord}_x Z.$$

**Proposition 2.9** ([Ein97]). Assume that  $G$  is critical at  $x$  and  $Z$  is the critical variety of  $G$  at  $x$ . If  $\text{def}_x(G) \geq 1$ , then  $\text{def}_x(G|_H) = \text{def}_x(G) - 1$  and  $G|_H$  is critical at  $x$ , where  $H$  is a general hypersurface in  $X$  passing through  $x$ . In particular,  $\text{def}_x(G) \leq \dim Z$ .

We also note that an implicit upper bound is given by the inequality in the following theorem.

**Theorem 2.10** ([Hel97]). Assume that  $G$  is critical at  $x$  and  $Z$  is the critical variety of  $G$  at  $x$ . Let  $e$  be the embedding dimension of  $Z$  at  $x$ ,  $d = \dim Z$  and  $m = \text{mult}_x Z$ . Then

$$m \leq \binom{e - \lceil \text{def}_x(G) \rceil}{e - d}.$$

In particular,

$$m \leq \binom{e - 1}{d - 1}.$$

*Remark 2.11.* Helmke (Example 3.5 [Hel99]) shows that when the critical variety  $Z$  is a hypersurface, then  $\text{def}_x(Z) = \dim X - \text{mult}_x Z$  which shows that the inequality in the above theorem is optimal.

Inspired by Theorem 2.10, we define an integer  $\alpha_{d,e}$  as follows and show that it is an upper bound of the deficit function.

**Definition 2.12.** Assume that  $G$  is critical at  $x$  and  $Z$  is the critical variety of  $G$  at  $x$ . Let  $e$  be the embedding dimension of  $Z$  at  $x$ ,  $d = \dim Z$  and  $m = \text{mult}_x Z$ . We define  $\beta_{d,e}(m) \leq d$  to be largest solution of  $y$  in the equation

$$\binom{e - y}{e - d} = m.$$

Denote by  $\alpha_{d,e}(m)$  the largest integer less than or equal to  $\beta_{d,e}(m)$ .

**Corollary 2.13.** Assume that  $G$  is critical at  $x$  and  $Z$  is the critical variety of  $G$  at  $x$ . Let  $e$  be the embedding dimension of  $Z$  at  $x$ ,  $d = \dim Z$  and  $m = \text{mult}_x Z$ . Then  $\text{def}_x(G) \leq \alpha_{d,e}(m)$ .

*Proof.* We note that

$$\binom{e-y}{e-d} - m$$

is a decreasing function of  $y$ . Therefore, by Theorem 2.10, we see that

$$\text{def}_x(G) \leq \alpha_{d,e}(m).$$

□

It is clear that  $\alpha_{d,e}(m) \leq d-1$  if  $m = \text{mult}_x Z \geq 2$ . In fact, we know that the larger the number  $m$  the smaller the integer  $\alpha_{d,e}(m)$ .

The above corollary together with Proposition 2.8 (1) implies one of the practically useful results of the paper.

**Lemma 2.14.** *Let  $X$  be a smooth projective variety of dimension  $n$  and  $x \in X$  a point. Assume that  $L$  is an ample line bundle on  $X$ . Let  $G$  be an effective  $\mathbb{Q}$ -divisor linearly equivalent to  $\lambda L$  for  $\lambda < 1$  and critical at  $x$ . If  $\text{ord}_x G = \lambda\sigma$  for some  $\sigma > n$ , then we have*

$$\frac{\text{def}_x(G)}{1-\lambda} \leq \frac{\sigma\alpha_{d,e}(m)}{\sigma-n+\alpha_{d,e}(m)},$$

where  $m = \text{mult}_x Z(G)$ .

Moreover, if  $L^n > \sigma_n^n \geq n^n$ , then there exist an effective  $\mathbb{Q}$ -divisor  $G$  linearly equivalent to  $\lambda L$  for  $\lambda < 1$  and critical at  $x$  with

$$\frac{\text{def}_x(G)}{1-\lambda} < \frac{\sigma_n\alpha_{d,e}(m)}{\sigma_n-n+\alpha_{d,e}(m)}.$$

*Proof.* By Proposition 2.8 and Corollary 2.13, we have the following inequality

$$\text{def}_x(G) \leq \min\{\alpha_{d,e}(m), n - \text{ord}_x G\}.$$

If  $\alpha_{d,e}(m) \leq n - \text{ord}_x G = n - \lambda\sigma$ , then

$$1 - \lambda \geq \frac{\sigma - n + \alpha_{d,e}(m)}{\sigma}.$$

Therefore,

$$\frac{\text{def}_x(G)}{1-\lambda} \leq \frac{\alpha_{d,e}(m)}{1-\lambda} \leq \frac{\sigma\alpha_{d,e}(m)}{\sigma-n+\alpha_{d,e}(m)}.$$

If  $\alpha_{d,e}(m) \geq n - \text{ord}_x G = n - \lambda\sigma$ , then  $\lambda \geq \frac{n-\alpha_{d,e}(m)}{\sigma}$ . Therefore,

$$\frac{\text{def}_x(G)}{1-\lambda} \leq \frac{n-\lambda\sigma}{1-\lambda} \leq \frac{\alpha_{d,e}(m)}{1-\frac{n-\alpha_{d,e}(m)}{\sigma}} = \frac{\sigma\alpha_{d,e}(m)}{\sigma-n+\alpha_{d,e}(m)}.$$

Now let  $\sigma'$  be a positive number such that  $\sqrt[n]{L^n} > \sigma' > \sigma_n$ . Since  $L^n > (\sigma')^n$ , then there is an effective  $\mathbb{Q}$ -divisor  $D$  linearly equivalent to  $L$  such that  $\text{ord}_x D > \sigma'$ . Let  $c$  be the log canonical threshold  $\text{lct}_x(X, D)$  at  $x$  and  $G' = cD$ . By applying tie breaking to  $G'$ , we may find a  $\mathbb{Q}$ -divisor  $G$  linearly equivalent to  $\lambda L$  with



$|\lambda - c| \ll 1$  such that  $G$  is critical at  $x$  and  $\text{ord}_x G = \lambda\sigma$  with  $\sigma > \sigma_n$ . Since  $\frac{\sigma\alpha_{d,e}(m)}{\sigma-n+\alpha_{d,e}(m)}$  is a decreasing function of  $\sigma$ , we have the strict inequality

$$\frac{\text{def}_x(G)}{1-\lambda} \leq \frac{\sigma\alpha_{d,e}(m)}{\sigma-n+\alpha_{d,e}(m)} < \frac{\sigma_n\alpha_{d,e}(m)}{\sigma_n-n+\alpha_{d,e}(m)}.$$

□

The same idea used in the above proof together with Lemma 2.6 leads to the following result for the second step reduction in a special situation.

**Proposition 2.15.** *Under the assumption in Lemma 2.14, assume in addition that the critical variety  $Z(G)$  is smooth at  $x$  and  $L^d \cdot Z(G) \geq \sigma_n^d$ , where  $d = \dim Z(G)$ . Then there is an effective divisor  $G_1$  linearly equivalent to  $\lambda_1 L$  with  $\lambda_1 < 1$  such that  $G_1$  is critical at  $x$  and  $d' = \dim Z(G_1) < \dim Z(G)$ . Moreover,*

$$\frac{\text{def}_x G_1}{1-\lambda_1} < \frac{\sigma_n \alpha'}{\sigma_n - n + \alpha'},$$

where  $m' = \text{mult}_x Z(G_1)$  and  $\alpha' = \alpha_{d',e'}(m')$ .

*Proof.* By our assumption, we can pick a sufficiently small  $\varepsilon_1$  such that  $\varepsilon_1 < \min\{1, \frac{\lambda(\sigma-\sigma_n)}{4}\}$  and an effective  $\mathbb{Q}$ -divisor  $D$  linearly equivalent to  $(1-\lambda)L$  with

$$\text{ord}_x D|_Z = (1-\lambda)(\sigma_n - \varepsilon_1) > \text{def}_x(G).$$

Let  $\varepsilon_2$  be a sufficiently small positive number such that  $\varepsilon_2 < \frac{\varepsilon_1}{3}$  and  $t < \min\{\frac{1}{2}, \lambda\}$  be a sufficiently small positive number such that

$$\text{def}_x((1-t)G) < \text{def}_x(G) + \varepsilon_2.$$

By Lemma 2.6, we can find a new divisor  $G_1 = (1-t)G + G'$  linearly equivalent to  $\lambda_1 L$  with  $\text{mult}_x G'|_Z \geq s'(1-\lambda)(\sigma_n - \varepsilon_1) + \delta$  for some  $0 < s' < 1$  such that  $G_1$  is critical at  $x$  and  $d' = \dim Z(G_1) < \dim Z(G)$ , where  $\delta$  is a sufficiently small positive number such that  $\delta < \frac{\varepsilon_1}{6\sigma_n}$  and  $\lambda_1 = (1-t)\lambda + s'(1-\lambda) + \delta < 1$ . Moreover,  $\text{def}_x G_1 \leq \min\{\alpha', \text{def}_x(G) + \varepsilon_2 - \text{ord}_x G'|_Z\}$ , where  $m' = \text{mult}_x Z(G_1)$  and  $\alpha' = \alpha_{d',e'}(m')$ . Notice that

$$\begin{aligned} \text{def}_x(G) + \varepsilon_2 - \text{ord}_x G'|_Z &\leq n - (1-t)\text{ord}_x G - \text{ord}_x G'|_Z + \varepsilon_2 \\ &< n - ((1-t)\lambda\sigma + s'(1-\lambda)(\sigma_n - \varepsilon_1) + \delta - \varepsilon_2) \\ &< n - \lambda_1(\sigma_n + \varepsilon_1) - ((1-t)\varepsilon_1 + \delta - \delta(\sigma_n + \varepsilon_1) - \varepsilon_2) \\ &< n - \lambda_1(\sigma_n + \varepsilon_1). \end{aligned}$$

The same argument as in the proof of Lemma 2.14 shows that

$$\frac{\text{def}_x G_1}{1-\lambda_1} < \frac{\sigma_n \alpha'}{\sigma_n - n + \alpha'}.$$

□



It will be very helpful to know the singularities of  $Z$ , especially the multiplicity  $\text{mult}_x Z$ . In [Kaw97], Kawamata initiated the study of subadjunction formulae. For critical varieties, now we have the following characterization on their singularities.

**Theorem 2.16** ([FG12]). *Assume that  $G$  is critical at  $x$  with the critical variety  $Z$ . Then there exist an effective  $\mathbb{Q}$ -divisor  $D_Z$  on  $Z$  such that*

$$(K_X + D)|_Z = K_Z + D_Z$$

*and the pair  $(W, D_W)$  is klt at  $x$ . In particular,  $Z$  has at most rational singularity at  $x$ .*

*Remark 2.17.* Let  $Z$  be a critical variety and  $x \in Z$  be a point. Assume that  $\dim Z = 2$ . Since  $p$  is rational, then  $\text{mult}_x Z = e - 1$ , where  $e$  is the embedding dimension of  $Z$  at  $x$ .

### 3. VOLUMES OF GRADED LINEAR SERIES AND APPLICATIONS

Let  $X$  be a smooth projective variety of dimension  $n$ ,  $\mathcal{I}$  be an ideal sheaf of  $\mathcal{O}_X$  and  $L$  be an ample line bundle on  $X$ . Let  $q$  be a nonnegative real number. Then  $A_k^{qk} := \{H^0(X, kL \otimes \mathcal{I}^{\lceil qk \rceil})\}$  is a graded sublinear series. We define the *volume function* of this sublinear series by  $\text{Vol}(\mathcal{I}, q, L) := \lim_k \left\{ \frac{n!}{k^n} \dim A_k^{qk} \right\}$ . We may omit  $\mathcal{I}$  when it is the maximal ideal  $\mathfrak{m}_x$  of a point  $x \in X$ . For any two real numbers  $\beta \leq \gamma$ , we write  $\text{Vol}(\beta, \gamma, L)$  for the difference  $\text{Vol}(\beta, L) - \text{Vol}(\gamma, L)$ . We denote by  $F_k(q)$  the fixed part of  $|kL \otimes \mathfrak{m}_x^{\lceil kq \rceil}|$  and define  $\phi_k(q) = q - \frac{\text{ord}_x F_k(q)}{k}$  if  $|kL \otimes \mathfrak{m}_x^{\lceil kq \rceil}| \neq \emptyset$  and  $\phi_k(q) = -\infty$  otherwise. The function  $\phi(q) = \sup_k \{\phi_k(q)\}$  is called the *mobility in codimension one* of  $L$  at  $x$ . We can check that  $\phi(q) = \lim_k \{\phi_k(q)\}$ . One very useful property of  $\phi(q)$  is the following.

**Proposition 3.1** ([Hel99]). *The function  $\phi(q)$  is a concave down function on  $\mathbb{R}_{\geq 0}$ .*

The following result, the idea of its proof is due to Fujita (see [Fuj93]), has appeared in [Kaw97], [Hel99] and [Lee99] in different looks.

**Proposition 3.2.** *Let  $\beta, \gamma \in \mathbb{R}_{\geq 0}$ . Assume that  $\text{Vol}(t, L) \geq 0$  for  $\gamma \geq t \geq \beta$ .*

$$\text{Vol}(\beta, \gamma, L) \leq n \int_{\beta}^{\gamma} \phi(t)^{n-1} dt.$$

*Proof.* Let  $k$  be a sufficiently divisible integer and  $t$  be a rational number. We may assume that  $kt$  is an integer, denoted by  $j$ . Since  $F_k(j)$  is the fixed part of  $|kL \otimes \mathfrak{m}_x^j|$ , then  $h^0(kL \otimes \mathfrak{m}_x^j) = h^0((kL - F_k(j)) \otimes \mathfrak{m}_x^j)$ . Let  $\pi : Y \rightarrow X$  be the blowing up at  $x$ ,  $E$  be the exceptional divisor, and  $\widetilde{F}_k(j)$  be the strict transform

of  $F_k(j)$ . Then  $\pi^*(F_k(j)) = \widetilde{F_k(j)} + \text{ord}_x F_k(j)E$  and

$$\begin{aligned} h^0(Y, \pi^*(kL) - jE) &= h^0(Y, \pi^*(kL - F_k(j)) - jE) \\ &= h^0(Y, \pi^*(kL - F_k(j)) + \text{ord}_x F_k(j)E - jE). \end{aligned}$$

Then

$$\begin{aligned} &h^0(Y, \pi^*(kL) - jE) - h^0(Y, \pi^*(kL) - (j+1)E) \\ &\leq \dim \text{Im}(H^0(Y, \pi^*(kL) - jE) \rightarrow H^0(E, (\pi^*(kL) - jE)|_E)) \\ &= \dim \text{Im}(H^0(Y, \pi^*(kL - F_k(j)) - jE) \rightarrow H^0(E, (\pi^*(kL) - jE)|_E)) \\ &= \dim \text{Im}(H^0(Y, \pi^*(kL - F_k(j)) + \text{ord}_x F_k(j)E - jE) \\ &\quad \rightarrow H^0(E, (\pi^*(kL - F_k(j)) + \text{ord}_x F_k(j)E - jE)|_E)) \\ &\leq h^0(\mathbb{CP}^{n-1}, \mathcal{O}_{\mathbb{CP}^{n-1}}(j - \text{ord}_x F_k(j))) \\ &\leq \frac{(j - \text{ord}_x F_k(j))^{n-1}}{(n-1)!}. \end{aligned}$$

Therefore

$$\text{Vol}(\beta, \gamma, L) \leq \lim_{k \rightarrow \infty} \frac{\sum_{j=[k\beta]+1}^{[k\gamma]} (j - \text{ord}_x F_k(j))^{n-1}}{\frac{(n-1)!}{\frac{k^n}{n!}}} = n \int_{\beta}^{\gamma} \phi(t)^{n-1} dt.$$

□

The idea of the proofs of the following two propositions comes essentially from [Hel99].

**Proposition 3.3.** *Let  $L$  be an ample line bundle on  $X$  with  $L^n > \sigma^n \geq n^n$  and  $x \in X$ . Assume that for some  $q > \sigma$  the linear system  $|kL \otimes \mathfrak{m}_x^{kq}|$  is nonempty for a sufficiently large  $k$ . There exists an effective  $\mathbb{Q}$ -divisor  $G$  linearly equivalent to  $\lambda L$  for some positive  $\lambda < 1$  such that it is critical at  $x$  with  $\text{ord}_x G > \lambda\sigma$ . Furthermore, if the critical variety  $Z = Z(G)$  is a divisor in  $X$  with  $\text{mult}_x Z = m$ . Then  $\frac{\text{def}_x G}{1-\lambda} < \frac{n-m}{1-\lambda} - \frac{\lambda}{1-\lambda} \phi_k(q)$ . In particular, if  $\phi(q) > (n-m) - \mu(q-\sigma)$  for some positive number  $\mu$ , then  $\frac{\text{def}_x G}{1-\lambda} < \frac{n-m+\mu\sigma}{1+\mu}$ .*

*Proof.* Let  $D$  be a general element in  $|kL \otimes \mathfrak{m}_x^{kq}|$ . Let  $c = \text{lct}(X, D/k) < 1$  and  $G' = \frac{c}{k}D$ . Hence  $G'$  is linearly equivalent to  $cL$  with  $\text{ord}_x G' \geq cq$ . If  $G'$  is critical at  $x$ , then we let  $G = G'$  and  $\lambda = c$ . Otherwise, we can perturb  $G'$  to get the desired divisor  $G$ . By tie breaking, for any  $0 \leq \alpha \ll 1$  and some  $0 \leq \epsilon \ll 1$ , there is an effective  $\mathbb{Q}$ -divisor  $H$  linearly equivalent to  $\alpha G$  such that  $G = (1-\epsilon)(G' + H)$  is critical at  $x$ . Hence  $G$  is linearly equivalent to  $\lambda L$ , where  $\lambda = (1-\epsilon)(1+\alpha)c$ . When  $\alpha$  is sufficiently small, we have  $\lambda < 1$  and  $\text{ord}_x G \geq (1-\epsilon)\text{ord}_x G' \geq (1-\epsilon)cq > \lambda\sigma$ .

Now assume that the critical variety  $Z$  is a divisor. We write

$$G = (1 - \epsilon)(G' - \frac{c}{k}F_k(q)) + ((1 - \epsilon)(\frac{c}{k}F_k(q) + H)).$$

By 2.8 (3) we have

$$\begin{aligned} \text{def}_x(G) &= \text{def}_x^Z(G) \\ &\leq \text{def}_x^Z((1 - \epsilon)(\frac{c}{k}F_k(q) + H)) - \text{ord}_x((1 - \epsilon)(G' - \frac{c}{k}F_k(q))) \\ (2) \quad &\leq \text{def}_x^Z(Z) - (1 - \epsilon)\text{ord}_x(G' - \frac{c}{k}F_k(q)) \\ &\leq \text{def}_x^Z - (1 - \epsilon)c\phi_k(q) \\ &\leq (n - m) - (1 - \epsilon)c\phi_k(q). \end{aligned}$$

Recall that  $\lambda = (1 - \epsilon)(1 + \alpha)c$ ,  $c < 1$  and  $0 \leq \alpha \ll 1$ . Then  $(1 + \alpha)c < 1$  and we obtain that

$$\begin{aligned} \frac{\text{def}_x(G)}{1 - \lambda} &\leq \frac{(n - m)}{1 - \lambda} - \frac{1}{(1 + \alpha)c} \frac{\lambda}{1 - \lambda} \phi_k(q) \\ &< \frac{(n - m)}{1 - \lambda} - \frac{\lambda}{1 - \lambda} \phi_k(q). \end{aligned}$$

Hence we obtain the first inequality. By the hypothesis on  $\phi(q)$ , there exist  $k$  sufficiently large such that  $\phi_k(q) > (n - m) - \mu(q - \sigma)$ . We obtain

$$(3) \quad \text{def}_x(G) < (1 - \lambda)(n - m) + \lambda\mu(q - \sigma).$$

Note that by 2.8 (1) we have  $\text{def}_x(G) \leq n - \lambda q \leq \sigma - \lambda q$ . Then

$$\frac{\lambda}{1 - \lambda}(q - \sigma) \leq \sigma - \frac{\text{def}_x(G)}{1 - \lambda}.$$

Therefore, from the inequality (3), we get

$$p = \frac{\text{def}_x G}{1 - \lambda} < (n - m) + \mu(\sigma - p).$$

Solving for  $p$  from this inequality, we see that

$$p < \frac{n - m + \mu\sigma}{1 + \mu},$$

□

**Proposition 3.4.** Assume that  $L^n > \sigma^n \geq n^n$ . For a real number  $w$  with  $0 \leq w < n - 1$ , we set  $\mu(w)$  be the minimal positivity number satisfying

$$(4) \quad \frac{(\frac{w}{\sigma} + \mu)^n}{\mu(1 + \mu)^{n-1}} \leq 1.$$

There exist a rational number  $q > \sigma$  such that

$$(5) \quad \phi(q) > w - \mu(w)(q - \sigma)$$

for all numbers  $w \in [0, n-1)$ . In particular, there exists a rational number  $q > \sigma$  such that for all  $w \in [0, n-1)$ , we have

$$\phi(q) > w - \frac{w}{\sigma(\sigma-1-w)}(q-\sigma).$$

*Proof.* Assume contrarily that for every rational number  $q > \sigma$ , there exist some  $w$  such that (5) fails. For simplicity, we denote by  $l_w(q) = w - \mu(w)(q - \sigma)$ . We claim that then there is a real number  $w \in [0, n-1)$  such that for any rational number  $q > \sigma$ , we have  $\phi(q) \leq l_w(q)$ .

We define  $\psi(q) := \sup_{w \in [0, n-1)} \{l_w(q)\}$  for  $q \geq \sigma$ . As a supreme of a family of linear functions, the function  $\psi(q)$  is a concave up function of  $q$  with  $\psi(\sigma) = n-1$ . By our assumption,  $\psi(q) \geq \phi(q)$  for all  $q > \sigma$ . Since  $\phi(q)$  is concave down and  $\psi(q)$  is concave up for  $q \geq \sigma$ , this implies that  $\phi(q)$  and  $\psi(q)$  is separated by a tangent line of  $\psi(q)$ . Hence there is a  $w \in [0, n-1)$  such that for every  $q \geq \sigma$ , we have  $\phi(q) \leq l_w(q)$ . Now since that  $\phi(q)$  is a concave down function and for any  $\nu \geq \mu(w)$ , we have

$$\frac{(\frac{w}{\sigma} + \nu)^n}{\nu(1 + \nu)^{n-1}} < 1.$$

By increasing  $\mu(w)$  to a larger number  $\nu$  if it is necessary, we may assume that

$$\phi(q) \leq w - \nu(q - \sigma) := l(q)$$

for any  $q \geq 0$ .

Let  $a = \frac{w+\nu\sigma}{1+\nu}$  and  $b = \sigma + \frac{w}{\nu}$ . It is easy to check that  $l(a) = a \leq \sigma$ ,  $l(b) = 0$  and  $b \geq \sigma$ . By Proposition 3.2, we have

$$\begin{aligned} \text{Vol}(0, b, L) &\leq \int_0^a nt^{n-1}dt + \int_a^b n\phi(t)^{n-1}dt \\ &\leq \int_0^a nt^{n-1}dt + \int_a^b nl(t)^{n-1}dt \\ &= a^n - \frac{1}{\nu}[l(b)^n - l(a)^n] \\ &= a^n(1 + \frac{1}{\nu}) = \sigma^n \frac{(\frac{w}{\sigma} + \nu)^n}{\nu(1 + \nu)^{n-1}}. \\ &\leq \sigma^n. \end{aligned}$$

Consequently,  $\text{Vol}(\mathbf{m}_x, b, L) \geq \text{Vol}(\mathbf{m}_x, 0, L) - \sigma^n = L^n - \sigma^n > 0$ . It follows that  $|kL \otimes \mathbf{m}_x^{rk}| \neq \emptyset$  for a rational number  $r > b$  and  $k \gg 0$  which implies that  $\phi(r) \geq 0$ . However, we have  $\phi(r) \leq l(r) < 0$ . This is a contradiction.

For the last assertion, it is enough to check that  $\frac{w}{\sigma(\sigma-1-w)}$  satisfying the inequality (4). □

**Corollary 3.5.** *Under the assumptions in Proposition 3.3, we have*

$$\frac{\text{def}_x(G)}{1-c} < \frac{\sigma(n-m)}{\sigma-1}.$$

#### 4. GLOBAL GENERATION OF ADJOINT LINE BUNDLES

By Proposition 2.7, in order to get smaller lower bounds on  $L^d \cdot Z$ , we want smaller upper bounds for  $(\frac{\text{def}_x(G)}{1-\lambda})^{n-1} \cdot \text{mult}_x Z$ . By Lemma 2.14, we see that the integer  $\alpha_{d,e}(m)$  is a key factor. In lower dimensional case, one can easily find  $\alpha_{d,e}(m)$  by solving the equation in Definition 2.12.

*Example 4.1.* Assume that  $Z$  is the critical variety of an effective divisor  $G$  at  $x$  and  $\dim Z = 2$ . By Remark 2.17, we know that  $m = \text{mult}_x Z = e - 1$ , where  $e$  is the embedding dimension. Then by Theorem 2.10, we have the following inequality

$$m \leq \binom{m+1-\alpha_{2,e}(m)}{m-1}.$$

If  $m \geq 2$ , then  $\text{def}_p(G) \leq \alpha_{2,e}(m) = 1$ .

*Example 4.2.* Let  $G$  be an effective divisor on a smooth projective variety  $X$  of dimension 5. Assume that  $Z(G)$  is a critical variety of dimension 3 and  $x \in Z(G)$  is a point. If  $Z(G)$  is not smooth at  $x$ , then

$$\text{def}_x(G) \leq \alpha_{3,5}(m) = \begin{cases} 2, & \text{mult}_x Z(G) = 2, 3; \\ 1, & \text{mult}_x Z(G) = 4, 5, 6. \end{cases}$$

By calculating the integer  $\alpha_{d,e}(m)$  and applying Lemma 2.14, Corollary 3.5 and Proposition 2.7, we can easily prove the following effective result on global generation of adjoint line bundles on 5-folds. In the proof, we will fix  $d$  in each case, therefore we will use  $\alpha(m)$  for  $\alpha_{d,e}(m)$ . We can check that  $\alpha(m) \leq \alpha_{d,n}(m)$ .

**Theorem 4.3.** *Let  $X$  be a smooth projective variety of dimension 5,  $L$  be an ample divisor on  $X$  and  $x \in X$  be a point. Assume that  $\sqrt[5]{L^5} > 7$  and  $\sqrt[d]{L^d \cdot Z} \geq 7$  for any subscheme  $Z \subset X$  containing  $x$  of dimension  $\dim Z = d$ , where  $d = 1, 2, 3, 4$ . Then  $\mathcal{O}_X(K_X + L)$  is globally generated at  $x$ .*

*Proof.* Let  $\sigma = 7$ . By Proposition 3.4, there exist a rational number  $q > 7$  such that  $\phi(q) > w - \mu(w)(q - \sigma)$  for all  $w \in [0, n - 1)$ . By proposition 3.3, we obtain an effective  $\mathbb{Q}$ -divisor  $G$  linearly equivalent to  $\lambda L$  with  $\lambda < 1$  and it is critical at  $x$  with multiplicity  $\text{ord}_x G > \lambda \sigma$ . Let  $Z$  be the critical variety and  $m = \text{mult}_x Z$  be the multiplicity. If  $\dim Z = 0$ , then the theorem follows from Lemma 2.1.

(1) Assume that  $\dim Z = 1$ . Then  $Z$  is smooth and  $\alpha(m) = 1$ , therefore

$$\frac{\text{def}_x(G)}{1-\lambda} < \frac{\sigma}{\sigma-5+1} \leq \frac{7}{3} < 7.$$

Applying Proposition 2.7 and Lemma 2.1, we know that the conclusion is true.

- (2) Assume that  $\dim Z = 2$ . Then  $m = \text{mult}_x Z \leq 4$ . If  $m \geq 2$ , then  $\alpha(m) = 1$  and

$$\frac{\text{def}_x(G)}{1 - \lambda} < \frac{\sigma}{\sigma - 5 + 1} \leq \frac{7}{3} < \frac{7}{\sqrt{m}}.$$

If  $m = 1$ , then  $\alpha(m) \leq 2$  and

$$\frac{\text{def}_x(G)}{1 - \lambda} < \frac{2\sigma}{\sigma - 5 + 2} \leq \frac{7}{2} < 7.$$

In both cases, we can construct a new divisor  $G_1$  with the following properties:  $G_1$  is linearly equivalent to  $\lambda_1 L$  for  $\lambda_1 < 1$ ;  $G_1$  critical at  $x$  with the critical variety  $Z_1$  properly contained in  $Z$ ;

$$\frac{\text{def}_x(G_1)}{1 - \lambda_1} < \begin{cases} \frac{7}{3} & m \geq 2 \\ \frac{7}{2} & m = 1, \end{cases}$$

by Proposition 2.7. If  $Z_1$  is a point, then the theorem follows directly from Lemma 2.1. If  $Z_1$  is a curve, we apply Proposition 2.7 again and then Lemma 2.1.

- (3) Assume that  $\dim Z = 3$ . Then  $m \leq 6$ . If  $m \geq 2$ , then  $\alpha(m) \leq 2$  and

$$\frac{\text{def}_x(G)}{1 - \lambda} < \frac{2\sigma}{\sigma - 5 + 2} \leq \frac{7}{2} < \frac{7}{\sqrt[3]{m}}.$$

If  $m = 1$ , then  $\alpha(m) \leq 3$  and

$$\frac{\text{def}_x(G)}{1 - \lambda} < \frac{3\sigma}{\sigma - 5 + 3} \leq \frac{21}{5} < 7.$$

In both cases, we can construct a new divisor  $G_1$  with the following properties:  $G_1$  is linearly equivalent to  $\lambda_1 L$  for  $\lambda_1 < 1$ ;  $G_1$  critical at  $x$  with the critical variety  $Z_1$  properly contained in  $Z$ ; and

$$\frac{\text{def}_x(G_1)}{1 - \lambda_1} < \begin{cases} \frac{7}{2} & m \geq 2 \\ \frac{21}{5} & m = 1 \end{cases}$$

by Proposition 2.7. We only need to take special care of the case that  $Z_1$  is a surface. Assume  $Z_1$  is a surface. Then

$$m_1 = \text{mult}_x Z_1 \leq \begin{cases} 4 & m \geq 2 \\ 2 & m = 1. \end{cases}$$

Consequently,

$$\frac{\text{def}_x(G_1)}{1 - \lambda_1} < \frac{7}{\sqrt{m}}.$$

We apply Proposition 2.7 again to draw our conclusion.

- (4) Assume that  $\dim Z = 4$ . Then  $m \leq 4$  and  $\alpha(m) \leq 5 - m$ .

- (a) If  $m = 1$ , then by Proposition 2.15, we can construct a new divisor  $G_1$  critical at  $x$  with the critical variety  $Z_1$  properly contained in  $Z$  and

$$\frac{\text{def}_x(G_1)}{1 - \lambda_1} < \frac{\sigma\alpha'}{\sigma - 5 + \alpha'}.$$

Since  $m' = \text{mult}_x Z_1 \leq 3$ , following the arguments in cases (1), (2) and (3), we know that  $\frac{\text{def}_x(G_1)}{1 - \lambda_1} < \frac{7}{\sqrt{m'}}$ . Hence we can apply Proposition 2.7 again.

- (b) If  $m \geq 2$ , by Proposition 3.3 and Corollary 3.5 we know that

$$\frac{\text{def}_x(G)}{1 - \lambda} < \frac{3\sigma}{\sigma - 1} = \frac{7}{2} < \frac{7}{\sqrt[4]{4}}.$$

By Proposition 2.7, we can construct a new divisor  $G_1$  critical at  $x$  with the critical variety  $Z_1$  properly contained in  $Z$  and

$$\frac{\text{def}_x(G_1)}{1 - \lambda_1} < \frac{7}{2} \leq \min\left\{7, \frac{7}{\sqrt{4}}, \frac{7}{\sqrt[3]{6}}\right\}.$$

Then we can apply Proposition 2.7 again till we get a 0-dimensional  $Z$ .

Summarizing our argument, we see that  $\mathcal{O}_X(K_X + L)$  is globally generated at  $x$ .

□

*Remark 4.4.* We note that the same argument as in the proof of Theorem 4.3 will give a concise proof of Kawamata's result (Theorem 1.6) on 4-folds.

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